Math 33A Worksheet Week 5 Solutions

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Exercise 1. Determine whether the following sets of vectors are linearly independent or linearly dependent:

(a) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

(a) Any 3 vectors in a two dimensional space must be linearly dependent, so they are linearly dependent.

(b) This is linearly dependent since $1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$. Notice that any set of vectors including zero must be linearly dependent.

(c) the matrix with columns corresponding to these 4 vectors is upper triangular and has determinant $1 * 3 * 4 * 5 \neq 0$, and so these 4 vectors are linearly independent.

Exercise 2. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $\begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 0 \\ -8 & 0 & 2 \end{bmatrix}$. Find a basis for $\ker A$. Find a basis for $\mathrm{Im} A$.

First we row reduce A, yielding

$$\begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the solutions to $Ax = 0$ ($\ker A$) are given by $x_1 = x_3/4$, $x_2 = 0$ and x_3 free, so

$$\Big(\text{solutions to } (Ax = 0)\Big) = \ker A = \left\{ \begin{bmatrix} 1/4x_3 \\ 0 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbb{R} \right\} = \left\{ x_3 \begin{bmatrix} 1/4 \\ 0 \\ 1 \end{bmatrix} \mid x_3 \in \mathbb{R} \right\}$$

so $\left\{ \begin{bmatrix} 1/4 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\ker A$.

The first and second columns of the row reduced echelon form of A are the columns with a leading one, so the first and second columns of A form a basis for $\operatorname{im} A$:

$$\left\{ \begin{bmatrix} 4 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Exercise 3. Find a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies each of the following conditions, or explain why such a linear transformation doesn't exist:

(a) $\ker A = \{\vec{0}\}$, $\operatorname{im} A = \{\vec{0}\}$.

(b) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}\}$ and $\det A = 0$

(c) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}\}$ and $\det A \neq 0$

(d) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$, $\operatorname{im} A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$.

(a) If $\operatorname{im} A = \{\vec{0}\}$, then $\dim \operatorname{im} A = 0$. If $\ker A = \{\vec{0}\}$, then $\dim \ker A = 0$. But since $\dim \operatorname{im} A + \dim \ker A = \dim \mathbb{R}^3 = 3$, it's impossible for both to be satisfied at the same time.

(b) Since $\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \in \ker A$, we have $x_1 = -x_3/3$, $x_2 = -x_3$, x_3 free as a solution to the equation $Ax = 0$. Therefore, the matrix

$$A = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

has the desired kernel. A also automatically has determinant 0 since it is not invertible.

(c) If $\det A \neq 0$, then A is invertible and $\dim \ker A = 0$. Therefore, it is impossible for both conditions to be satisfied at the same time for a linear transformation A.

(d) Recall that $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the first column of A. Since $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \ker A$, $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$, so the first column of A must be zero. Similarly, the second column of A must be zero. In order for the image of A to be the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, its columns must must span $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so the following works:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise 4. Find the kernel of the following matrices:

(a) $\begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$

(c) A for $A : \mathbb{R}^n \to \mathbb{R}^n$ invertible.

(d) A with row reduced echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) RREF:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

so

$$\ker = \left\{ \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbb{R} \right\}$$

(b) RREF has a leading one in both columns, so $\ker = \{\vec{0}\}$.

(c) A linear invertible transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ has RREF equal to the identity matrix, so every column has a leading one, so $\ker A = \{\vec{0}\}$. Alternatively, since A is invertible, $Ax = 0$ has the unique solution $\vec{0}$.

Challenge/Conceptual Problems

Exercise 5.
Let A be a $m \times n$ matrix, $v \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ such that $Av = w$. Now suppose that v' is another vector in $\mathbb{R}^n$ such that $Av' = 0$, i.e. v' is in $\ker A$. What is $A(v + v')$?
(For a concrete example, Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $w = Av = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. Now if we let $v' = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so $Av' = 0$, what is $A(v + v') = A \begin{bmatrix} -1 \\ 2 \end{bmatrix}$?)

$A(v + v') = Av + Av' = w + 0 = w$, since A is a linear transformation.
Note the important part of this exercise is that the image of a vector is not changed when you add something from the kernel!

Exercise 6.
Let A be any matrix, and B be the RREF for A.

(a) Is $\ker A = \ker B$?

(b) Is $\operatorname{im} A = \operatorname{im} B$?

(a) The kernel of A is equal to the kernel of B! Note that the kernel of A is the same as the set of solutions to the matrix equation $Ax = 0$, which we set up as the augmented matrix equation $(A|0)$. We row reduce this to RREF and get the augmented matrix $(B|0)$, and so we are now solving the equation $Bx = 0$! Since row reduction doesn't change our solution set, the kernel of B will be the same as the kernel of A.
(b) No! The image of A and image of B will not necessarily be the same. Here's an example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, rref(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The matrix A has as its image the line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, i.e. the line $y = x$. However, its rref has the x-axis as its image.
However, the dimension of both images will be the same! The dimension of the column space is the number of pivots in the rref. Its important to note that while the dimensions of the images are equal, the actual images themselves are not equal.